Non-parametric identification of a memoryless system with a cascade structure

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The optimal cascade model of a system with a cascade structure is presented and a non-parametric identification procedure is introduced. Consistency of identification procedures is shown and examples of algorithms derived from Parzen kernel regression estimates are given.

1. Introduction and preliminaries

We identify a system with a cascade structure consisting of two memoryless and randomly disturbed subsystems. A model has the same structure as the system, i.e. it consists of two subsystems connected in a cascade. Our approach is non-parametric and we use classical and sequential Rosenblatt–Parzen kernel procedures in order to identify the system and we show basic asymptotical properties of the obtained algorithms.

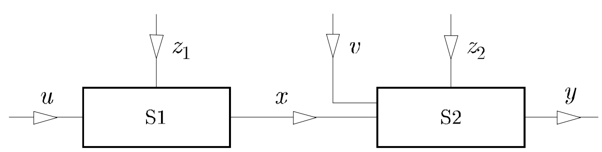

Figure 1. Identified complex system with cascade structure.

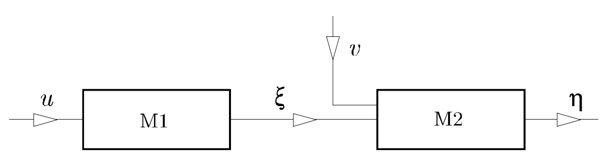

Figure 2. Cascade model of the identified system.

The system under consideration is shown in Fig. 1. Inputs u and v of subsystems $S1$ and $S2$ as well as noises z_1 and z_2 are random. It is assumed that vectors u and v are of the same dimension, say p. By U, V, X, Y we denote appropriate input and output random variables. The model of the system, see Fig. 2, consists of two submodels $M1$ and $M2$ and its quality is

$$Q = E\|\xi - X\|^2 + E\|\eta - Y\|^2,$$

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where $\|.\|$ is the euclidean norm. We shall now show that in the optimal model

$$\xi = \Phi(u) \tag{1.1}$$

and

$$\eta = \Psi(\Phi^{-1}(\xi), v) \tag{1.2}$$

where

$$\Phi(u) = E\{X|U = u\} \tag{1.3}$$

and

$$\Psi(u, v) = E\{Y|U = u, V = v\} \tag{1.4}$$

and Φ^{-1} is – assumed to exist – the inverse of Φ.

In order to show that (1.1) and (1.2) define the optimal models $M1$ and $M2$ we shall consider a model shown in Fig. 3. Let us take into account a quadratic quality index

$$Q_1 = E\|\zeta - X\|^2 + E\|\kappa - Y\|^2.$$

Obviously

$$\min Q_1 \leq \min Q, \tag{1.5}$$

where $\min Q$ and $\min Q_1$ are minimal values of quality indexes Q and Q_1, respectively. Clearly, for the optimal model $M3$

$$\zeta = \Phi(u) \text{ and } \kappa = \Psi(u, v).$$

Now it can be easily verified that for submodels $M1$ and $M2$ defined by (1.1) and (1.2) and the optimal model $M3$

$$\xi = \zeta \text{ and } \eta = \kappa.$$

Hence, by (1.5), submodels of (1.1) and (1.2) are optimal.

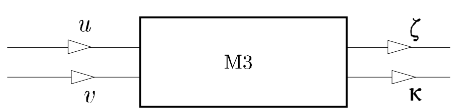

Figure 3. Multivariate model of the identified system.

We assume that the probability distributions of input and output signals are not known and submodels (1.1) and (1.2) can be estimated from a sequence of independent observations $(U_1, V_1, X_1, Y_1), \ldots, (U_n, V_n, X_n, Y_n)$ of the random vector (U, V, X, Y).

We have to solve two basic problems, namely, to estimate regression functions Φ and Ψ and to estimate the inversion of the regression function Φ. The first problem, i.e. regression estimation, is extensively treated in the literature, whereas estimation of the regression inversion, to the authors' knowledge, has not been studied. Because of that, in the paper, a method to construct an estimate of the inversion of regression is proposed and its consistency is proved. As a result, we can solve the identification problem, i.e. we can estimate both the optimal models $M1$ and $M2$.

2. Submodel $M1$

Regression (1.1) can be rewritten in the following form :

$$\Phi(u) = \frac{g(u)}{f(u)}, \tag{2.1}$$

where f is the marginal density of U and $g(u) = \Phi(u)f(u)$. In order to estimate the optimal model $M1$, i.e. regression (2.1) we apply either the Rosenblatt-Parzen estimate

$$\widehat{\Phi}_n(u) = \frac{\sum_{i=1}^{n} X_i K\left(\frac{u-U_i}{h(n)}\right)}{\sum_{i=1}^{n} K\left(\frac{u-U_i}{h(n)}\right)} \tag{2.2}$$

or its sequential version

$$\widetilde{\Phi}_n(u) = \frac{\sum_{i=1}^{n} \frac{1}{h^p(i)} X_i K\left(\frac{u-U_i}{h(i)}\right)}{\sum_{i=1}^{n} \frac{1}{h^p(i)} K\left(\frac{u-U_i}{h(i)}\right)} \tag{2.3}$$

where K and $\{h(n)\}$ are suitably selected a. kernel and a sequence of numbers, respectively. Estimates (2.2) and (2.3) can be rewritten in the following form:

$$\widehat{\Phi}_n(u) = \frac{\widehat{g}(u)}{\widehat{f}(u)} \tag{2.4}$$

and

$$\widetilde{\Phi}_n(u) = \frac{\widetilde{g}(u)}{\widetilde{f}(u)} \tag{2.5}$$

where

$$\widehat{g}_n(u) = \frac{1}{nh^p(n)} \sum_{i=1}^{n} X_i K\left(\frac{u-U_i}{h(n)}\right) \tag{2.6}$$

$$\widetilde{g}_n(u) = \frac{1}{n} \sum_{i=1}^{n} \frac{1}{h^p(i)} X_i K\left(\frac{u-U_i}{h(n)}\right) \tag{2.7}$$

are estimates of $g(u)$, and

$$\widehat{f}_n(u) = \frac{1}{h^p(n)} \sum_{i=1}^{n} K\left(\frac{u-U_i}{h(n)}\right) \tag{2.8}$$

$$\widetilde{f}_n(u) = \frac{1}{n} \sum_{i=1}^{n} \frac{1}{h^p(i)} K\left(\frac{u-U_i}{h(n)}\right) \tag{2.9}$$

are estimates of $f(u)$.

The density estimate (2.8) was introduced by Rosenblatt (1956) while Parzen (1962) showed, for $p = 1$, its pointwise and uniform consistency. Cacoullos (1966) extended Parzen's result to multivariate densities. Regression estimate (2.2) was proposed by Nadaraya (1964) and developed by Noda (1976). Recursive density estimate (2.9) was suggested by Wolverton and Wagner (1969) whereas its basic properties, i.e. pointwise and uniform consistency, were studied by Yamato (1971). Ahmad and Lin (1976) as well as Noda (1976) examined regression estimate (2.3).

On the non-negative Borel kernel K we impose the following conditions:

$$\left.\begin{array}{c} \int K(u)du = 1,\ \ \int K^2(u)du < \infty \\ \\ \lim\limits_{\|u\| \to \infty} \|u\|^p K(u) = 0. \end{array}\right\} \tag{2.10}$$

The next two theorems are on the pointwise consistency of (2,2) and (2.3).

Theorem 1 (pointwise consistency of $\widehat{\Phi}_n$)
If $E\|X\| < \infty$, the kernel satisfies (2.10) and

$$h(n) \to 0, \tag{2.11}$$

$$nh^p(n) \to \infty \tag{2.12}$$

as $n \to \infty$, then

$$\widehat{\Phi}_n(u) \xrightarrow{p} \Phi(u)$$

as $n \to \infty$, at every point of continuity of both f and g.

Proof. The theorem can be proved by the same method as Theorems 1A and 2A in Parzen (1962) – for the case when the dimension of x is p, see Greblicki and Krzyżak (1978) – and will be omitted here. □

Theorem 2 (pointwise consistency of $\widetilde{\Phi}_n$)
If $E\|X\| < \infty$, the kernel satisfies (2.10), and , in addition to (2.11)

$$\frac{1}{n^2}\sum_{i=1}^{n}\frac{1}{h^p(i)} \to 0$$

as $n \to \infty$, then

$$\widetilde{\Phi}_n(u) \to \Phi(u)$$

as $n \to \infty$, at every point of continuity of both f and g.

Proof. The proof can be found in Greblicki and Krzyżak (1978). □

When $p = 1$, kernels

$$K(u) = \begin{cases} \frac{1}{2}, & \text{for } |u| \le 1 \\ 0, & \text{otherwise,} \end{cases} \tag{2.13}$$

$$K(u) = \begin{cases} 1 - |u|, & \text{for } |u| \le 1 \\ 0, & \text{otherwise,} \end{cases} \tag{2.14}$$

$$K(u) = \begin{cases} -\dfrac{16a^3}{9}u^2 + a, & \text{for } |u| \le \dfrac{3}{4a} \\ 0, & \text{otherwise,} \end{cases} \tag{2.15}$$

where a is an arbitrary positive number, can be used in our estimates. Greblicki and Krzyżak (1978) recommended a parabolic kernel (2.15) since it guarantees the best rate of the emergence of density estimates (2.8), (2.9) and estimates (2.6), (2.7).

The next two theorems on the uniform consistency will be useful in the next sections.

Theorem 3 (uniform consistency of $\widehat{\Phi}_n$)
If $E\|X\|^2 < \infty$, f and g are uniformly continuous, the kernel satisfies (2.10), and its Fourier transform k, where

$$k(t) = \int K(u)\exp(it'u)du,$$

and $'$ denotes transpose, is absolutely integrable i.e.

$$\int |k(t)|dt < \infty \tag{2.16}$$

and in addition to (2.11)

$$nh^{2p}(n) \to \infty \tag{2.17}$$

as $n \to \infty$, and, moreover, for some set $A \in R^p$,

$$\sup_{u\in A} \|\Phi(u)\| < \infty, \tag{2.18}$$

$$\inf_{u\in A} f(u) > 0, \tag{2.19}$$

then

$$\sup_{u\in A} \left\|\widehat{\Phi}_n(u) - \Phi(u)\right\| \xrightarrow{p} 0 \tag{2.20}$$

as $n \to \infty$.

Proof. Using the same method as in Theorem 3A of Parzen (1962) one may show that

$$\sup_{u\in A} \left\|\widehat{f}_n(u) - f(u)\right\| \xrightarrow{p} 0$$

as $n \to \infty$. Similarly it may be shown that

$$\sup_{u\in A} \|\widehat{g}_n(u) - g(u)\| \xrightarrow{p} 0$$

as $n \to \infty$. Application of Lemma 1 given below completes the proof. □

Lemma 1
Let Φ and f satisfy (2.18) and (2.19).If

$$\sup_{u\in A}\|g_n(u)-g(u)\|\overset{p}{\to}0$$

and

$$\sup_{u\in A}\|f_n(u)-f(u)\|\overset{p}{\to}0$$

as $n\to\infty$ then

$$\sup_{u\in A}\|\Phi_n(u)-\Phi(u)\|\overset{p}{\to}0$$

as $n\to\infty$, where $\Phi_n(u)=g_n(u)/f_n(u)$.

Remark 1
Assumption (2.16) is satisfied, e.g. for kernels (2.14) and (2.15) but not for window kernel (2.13).

Theorem 4 (uniform consistency of $\widehat{\Phi}_n$; Ahmad and Lin (1976))
If $E\|X\|^2<\infty$, the kernel satisfies (2.10) and its Fourier transform is absolutely integrable, and non-decreasing in the negative part and non-increasing in the positive part for each argument, and monotonically decreasing sequence $\{h(n)\}$ satisfies (2.11) and (2.17) and, moreover, (2.18) and (2.19) hold, then:

$$\sup_{u\in A}\left\|\widetilde{\Phi}_n(u)-\Phi(u)\right\|\overset{p}{\to}0$$

as $n\to\infty$.

3. Estimate of Ψ

The problem of consistency of an estimate of the optimal submodel $M2$ defined by (1.2) consists of two subproblems. The first is to estimate regression (1.4) while the other is to estimate the regression inversion Φ^{-1}. In order to estimate Ψ we can use either of the following two estimates:

$$\widehat{\Psi}_n(u)=\frac{\sum_{i=1}^{n}Y_iK\left(\frac{u-U_i}{h(n)}\right)K\left(\frac{v-V_i}{h(n)}\right)}{\sum_{i=1}^{n}K\left(\frac{u-U_i}{h(n)}\right)K\left(\frac{v-V_i}{h(n)}\right)} \tag{3.1}$$

or its sequential version

$$\widetilde{\Psi}_n(u)=\frac{\sum_{i=1}^{n}\frac{1}{h^q(i)}Y_iK\left(\frac{u-U_i}{h(i)}\right)K\left(\frac{v-V_i}{h(n)}\right)}{\sum_{i=1}^{n}\frac{1}{h^q(i)}K\left(\frac{u-U_i}{h(i)}\right)K\left(\frac{v-V_i}{h(n)}\right)} \tag{3.2}$$

where $q = p + s$, and s is the dimension of a vector v. For simplicity of notation, kernels in (3.1) and (3.2) are, like those in (2.2.) and (2.3), denoted by K. Asymptotical properties of these algorithms follow from Theorems 1–4.

4. Estimate of the regression inversion Φ^{-1}

We shall now consider a problem how to estimate the regression inversion. It should be mentioned that inversions of (2.2) and (2.3) may not exist despite the fact that the regression is invertible. In order to estimate Φ^{-1} we shall introduce a notion of pseudo-inversion.

Definition 1
A function ϕ^{+}on $\mathcal{Y}$ to $\mathcal{X}$ to is said to be a pseudo-inversion of a continuous function ϕ on $\mathcal{X}$ to $\mathcal{Y}$ if, for every $y \in \mathcal{Y}$, $\phi^{+}(y)$ is equal to any $x^{*} \in \mathcal{X}$ for which

$$\min_{x \in \mathcal{X}} \|\phi(x) - y\| = \|\phi(x^{*}) - y\|.$$

In the definition we admit also x's with infinite coordinates. Obviously, a pseudo-inversion depends on a norm, however for simplicity we use the euclidean one.

We shall now give some properties of a pseudo-inversion. Let $\mathcal{Y}_{\phi}$ be an image, by ϕ, of $\mathcal{X}$. Clearly, if ϕ has the inversion, $\phi^{-1}(y) = \phi^{+}(y)$ for every $y \in \mathcal{Y}_{\phi}$. Let $D^{y} = \{x : x \in \mathcal{X}, \phi(x) = y\}$. Obviously, $\phi^{+}(y) \in D^{y}$ for every $y \in \mathcal{Y}_{\phi}$. Moreover, if $\mathcal{X} = R$

$$\inf_{x \in D^{y}} x \leq \phi^{+}(y) \leq \sup_{x \in D^{y}} x.$$

For a scalar function ϕ defined on a bounded support,

$$\phi^{+}(y) = \min_{x \in D^{y}} x \tag{4.1}$$

and

$$\phi^{+}(y) = \max_{x \in D^{y}} x \tag{4.2}$$

for any $y \in \mathcal{Y}_{\phi}$, are examples of pseudo-inversions.

We shall now show that if a regression estimate Φ_n is uniformly consistent, then its pseudo-inversion is a pointwise consistent estimate of the inversion.

Theorem 5
Let A be a compact sot in R^{p} and let Φ be a function on A to R^{p} . If

$$\sup_{u \in A} \|\Phi_n(u) - \Phi(u)\| \xrightarrow{p} 0$$

as $n \to \infty$, then

$$\left\|\Phi_n^{+}(\xi) - \Phi^{-1}(\xi)\right\| \xrightarrow{p} 0 \tag{4.3}$$

as $n \to \infty$, at every $\xi \in R^{p}_{\Phi}$.

Proof. Fix $\xi \in R^p_\Phi$ and $u \in A$. First we shall show that for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\left| \|\Phi(u) - \xi\| - \|\Phi(u') - \xi\| \right| < \delta$$

implies

$$\|u - u'\| < \varepsilon.$$

If it were false, there would exist an $\varepsilon > 0$ and a sequence $\{u_n\}$ such that

$$\left| \|\Phi(u) - \xi\| - \|\Phi(u_n) - \xi\| \right| < \frac{1}{n}$$

and

$$\|u - u'\| \geq \varepsilon.$$

By the compactness of A, it follows from that, that there exists a point $u^* \in U$ and that $\Phi(u) = \Phi(u^*)$, which contradicts the assumption on existence of Φ^{-1}.

By the above, in order to prove (4.3) it suffices to show that

$$\left| \left\|\Phi\left(\Phi^{-1}(\xi)\right) - \xi\right\| - \left\|\Phi\left(\Phi_n^+(\xi)\right) - \xi\right\| \right| \stackrel{p}{\to} 0$$

as $n \to \infty$, at every $\xi \in R^p_\Phi$. The quantity in the above is not greater than

$$\begin{aligned}
&\left| \left\|\Phi\left(\Phi^{-1}(\xi)\right) - \xi\right\| - \left\|\Phi_n\left(\Phi_n^+(\xi)\right) - \xi\right\| \right| \\
&+ \left\|\Phi\left(\Phi_n^+(\xi)\right) - \xi\right\| - \left\|\Phi_n\left(\Phi_n^+(\xi)\right) - \xi\right\| \\
&\leq 2 \sup_{u \in A} \|\Phi(u) - \Phi_n(u)\| .
\end{aligned}$$

The last inequality is a consequence of the following two:

$$\begin{aligned}
&\left| \left\|\Phi\left(\Phi^{-1}(\xi)\right) - \xi\right\| - \left\|\Phi_n\left(\Phi_n^+(\xi)\right) - \xi\right\| \right| \\
&= \left| \inf_{u \in A} \|\Phi(u) - \xi\| - \inf_{u \in A} \|\Phi_n(u) - \xi\| \right| \\
&\leq \sup_{u \in A} \|\Phi_n(u) - \Phi(u)\| ,
\end{aligned}$$

and

$$\begin{aligned}
&\left| \left\|\Phi\left(\Phi_n^+(\xi)\right) - \xi\right\| - \left\|\Phi_n\left(\Phi_n^+(\xi)\right) - \xi\right\| \right| \\
&= \left\|\Phi\left(\Phi_n^+(\xi)\right) - \Phi_n\left(\Phi_n^+(\xi)\right)\right\| \leq \sup_{u \in A} \|\Phi_n(u) - \Phi(u)\| .
\end{aligned}$$

The proof is complete. □

5. Submodel $M2$

Our estimates of optimal model (1.2) are of the following forms:

$$\widehat{\Psi}_n\left(\widehat{\Phi}_n^+(\xi), v\right) \tag{5.1}$$

and

$$\widetilde{\Psi}_n\left(\widetilde{\Phi}_n^+(\xi), v\right), \tag{5.2}$$

where $\widehat{\Phi}_n^+$ and $\widetilde{\Phi}_n^+$ are pseudo-inversions of $\widehat{\Phi}_n$ and $\widetilde{\Phi}_n$, respectively. In order to show consistency of these estimates we need the following lemma.

Lemma 2
If ϕ is a continuous function on R^p to R^p and

$$\sup_x \|\phi(x) - \phi_n(x)\| \xrightarrow{p} 0 \tag{5.3}$$

and

$$X_n \xrightarrow{p} x_0 \tag{5.4}$$

as $n \to \infty$, then

$$\phi_n(X_n) \xrightarrow{p} \phi(x_0). \tag{5.5}$$

Proof. The proof is elementary since

$$\begin{aligned}\|\phi_n(X_n) - \phi(x_0)\| &\le \|\phi_n(X_n) - \phi(X_n)\| + \|\phi_n(X_n) - \phi(x_0)\| \\ &\le \sup_x \|\phi_n(x) - \phi_n(x)\| + \|\phi(X_n) - \phi(x_0)\|.\end{aligned}$$

□

Now we assume that a range of input v is a compact set B. Denote by D a cartesian product $A \times B$. By Theorem 5 and Lemma 2 we get a theorem on consistency of estimate $\Psi_n(\Phi_n^+(\xi), v)$ of $\Psi_n(\Phi_n(\xi), v)$.

Theorem 6
Let Ψ be continuous on D. If and

$$\sup_{(u,v)\in D} \|\Psi_n(u, v) - \Psi(u, v)\| \xrightarrow{p} 0$$

and

$$\sup_{u\in A} \|\Phi_n(u) - \Phi(u)\| \xrightarrow{p} 0$$

as $n \to \infty$, then

$$\Psi_n\left(\Phi_n^+(\xi), v\right) \xrightarrow{p} \Psi_n\left(\Phi_n^{-1}(\xi), v\right)$$

as $n \to \infty$, at every $\xi \in R_\Phi^p$ and $v \in B$.

Finally, by Theorems 3, 4 and 6 we get the next.

Theorem 7 (pointwise consistency of $\widehat{\Psi}_n(\widehat{\Phi}_n^+(.), .)$ and $\widetilde{\Psi}_n(\widetilde{\Phi}_n^+(.), .)$)
Let Φ and Ψ be continuous and bounded on A and D, respectively, and let the densities of U and V be bounded from zero on A and B, respectively, and let,

moreover, $E\|X\|^2 < \infty$, $E\|Y\|^2 < \infty$. If a continuous kernel and the number sequence satisfy appropriate assumptions of Theorem 3, then

$$\widehat{\Psi}_n\left(\widehat{\Phi}_n^+(\xi), v\right) \xrightarrow{p} \Psi_n\left(\Phi_n^{-1}(\xi), v\right)$$

as $n \to \infty$, at every $\xi \in R_\Phi^p$ and $v \in B$. If the kernel and the number sequence satisfy appropriate assumptions of Theorem 4, then

$$\widetilde{\Psi}_n\left(\widetilde{\Phi}_n^+(\xi), v\right) \xrightarrow{p} \Psi_n\left(\Phi_n^{-1}(\xi), v\right)$$

as $n \to \infty$, at every $\xi \in R_\Phi^p$ and $v \in B$.

Pseudo-inversions are not given explicitly (see, for example, examples (4.1) and (4.2) and consequently pseudo-inversions of estimates (2.2) and (2.3) have to be calculated numerically. Computational effort can be considerably decreased while using kernels with a bounded support, e.g. those in (2.14) and (2.15), since pseudo-inversions can be calculated piecewisely.

6. Window kernel

The problem of calculating of pseudo-inversions (2.2) and (2.3) is particularly simple, for $p = 1$, for an estimate

$$\Phi_n(u) = \frac{\overline{g}_n(u)}{\overline{f}_n(u)}, \tag{6.1}$$

where

$$\overline{f}_n(u) = \sum_{i=1}^{n} X_i K_1\left(\frac{u - U_i}{h(n)}\right) \tag{6.2}$$

and

$$\overline{g}_n(u) = \sum_{i=1}^{n} X_i K_2\left(\frac{u - U_i}{h(n)}\right) \tag{6.3}$$

and K_1 is a kernel (2.14) and K_2 is defined by (2.13). Since Fourier transform of K_2 is not absolutely integrable Theorem 3 is useless. However, by a method used by Parzen (1962) one can easily show that if the kernel and the sequence $\{h(n)\}$ satisfy assumptions of Theorem 3,

$$\sup_{u \in A} \|\overline{g}_n(u) - g(u)\| \xrightarrow{p} 0$$

as $n \to \infty$.

By a method used by Moore and Yackel (1977) in the proof of Theorem 2.1, one can show that if kernel and the sequence $\{h(n)\}$ satisfy assumptions of Theorem 3 then

$$\sup_{u \in A} \left\|\overline{f}_n(u) - f(u)\right\| \xrightarrow{p} 0$$

as $n \to \infty$. By the above we get the final theorem.

Theorem 8
Let Φ and the density of U satisfy conditions of Theorem 3 and, moreover, $E\|X\|^2 < \infty$. If K_1 and $\{h(n)\}$ satisfy assumptions of Theorem 3 and, moreover, K_2 is a window kernel (2.13), then

$$\sup_{u \in A} \left\| \overline{\Phi}_n(u) - \Phi(u) \right\| \xrightarrow{p} 0$$

as $n \to \infty$.

In order to prove Theorem 8 it suffices to apply Lemma 1.

7. Final remarks

The result of the paper can be easily extended to systems consisting of many subsystems connected in a cascade. Such problems arise while identifying a process of pollutants spreading in rivers or water canals.

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