Input nonlinearity recovering in a class of systems

Włodzimierz Greblicki

***Abstract*—A system in which a nonlinear memoryless part is followed by a nonlinear dynamic one is identified under nonparametric *a priori* information. The input nonlinearity is recovered with the kernel regression estimate.**

***Index Terms*—System identification, nonparametric identification, regression estimation, kernel estimation.**

I. Introduction

The problem of recovering the input nonlinearity in Hammerstein systems, i.e., systems in which a nonlinear memoryless subsystem is followed by a linear dynamic one, has received much attention so far, see, e.g., [1] for parametric, and [3], [7], [5], [9] or [4] for nonparametric algorithms, respectively. We deal with a nonparametric kernel estimate and show that it can be successfully applied for a larger class of systems, since we consider not only systems with linear transition matrices, i.e., Hammerstein ones, but also those with nonlinear matrices.

The problem is interesting also in applications, since some nonlinear processes in biology, [6], chemistry, [2] or medicine, [10], [11], are modelled with Hammerstein structures. We show that the input nonlinearity in such processes can be recovered even if their properties are more complicated, i.e., their dynamics is nonlinear.

II. Identification problem and algorithm

The system, see Fig. 1, consists of two nonlinear subsystems, a memoryless and dynamic ones. First has a characteristic m which means that $W_n = m(U_n)$. The dynamic part is described by the following nonlinear state equation:

$$\left.\begin{aligned} X_{n+1} &= A(X_n)X_n + bW_n \\ V_n &= c^T X_n, \end{aligned}\right\} \tag{1}$$

$n = 0, 1, \ldots$,with the initial vector $X_0 = 0$. The dimension k of the state vector X_n, the matrix $A(.)$, and vectors b, c are all unknown. Its output signal V_n is disturbed by noise Z_n and, therefore, $Y_n = V_n + Z_n$ is measured. We assume that $\{U_n; n = 0, 1, \ldots\}$ and $\{Z_n; n = 0, 1, \ldots\}$ are sequences of independent, identically distributed random variables. The sequences are mutually independent. Moreover, $E\{Z_n\} = 0$ and $\text{var}[Z_n] < \infty$. Our goal is to recover the input nonlinearity m from observations $(U_0, Y_0), \ldots, (U_n, Y_n)$ taken at the input and output of the whole system.

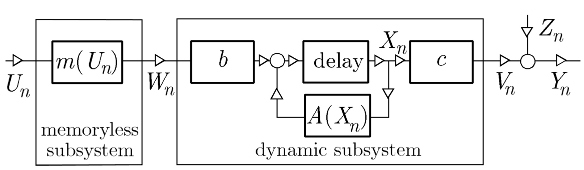

Fig. 1. The system.

The transition matrix $A(x)$, $x \in R^k$, represents the internal nonlinearity of the dynamic subsystem. For $A(x)$ independent of x, i.e., for $A(x) = A$, the subsystem is just linear and the whole system is reduced to a Hammerstein one. Such a case has received much attention in the literature, see [4] and references given therein. In this note, we show that the nonlinear input characteristic can be recovered even if the transition matrix of the dynamic part is nonlinear.

The author is with the Institute of Computer Engineering, Control, and Robotics, Wrocław University of Technology, S. Wyspiańskiego 27, 50-370 Wrocław, Poland.

On the system we impose additional restrictions. By f we denote the probability density of U_n's and assume that

$$f \text{ is an even function.} \tag{2}$$

The restriction is satisfied for, e.g., zero mean Gaussian, triangle, as well as uniform, distributions. Moreover,

$$m \text{ is an odd function.} \tag{3}$$

Owing to (2) and (3), the distribution of W_n's is symmetrical which means that W_n's and $-W_n$'s have identical distributions. In addition, we assume that

$$A(x) = A(-x) \tag{4}$$

for all $x \in R^k$.

Restrictions (2)-(4), imply

$$E\{Y_{n+1}|U_n = u\} = \alpha m(u) \tag{5}$$

with $\alpha = c^T b$. Since $E\{Y_{n+1}|W_n\} = c^T E\{A(X_n)X_n\} + c^T bW_n$, to verify (5), it suffices to show that $E\{A(X_n)X_n\} = 0$, for $n = 0, 1, 2, \ldots$. For $n = 0$, the equality is obvious, since $X_0 = 0$. Moreover, $X_1 = bW_0$ has a symmetrical distribution which, together with (4), implies a symmetrical distribution of $A(X_1)X_1$. In turn, since the distribution of bW_1 is symmetrical, so is that of $X_2 = A(X_1)X_1 + bW_1$. Thus, due to (4), $A(X_2)X_2$ is also distributed symmetrically. For similar reasons, $A(X_n)X_n$'s have all symmetrical distributions which implies the desired equality.

Due to (5), to recover $\alpha m(u)$, we estimate the regression $E\{Y_{n+1}|U_n = u\}$ and do it with the following kernel algorithm:

$$\hat{m}_n(u) = \sum_{i=0}^{n} Y_{i+1} K\left(\frac{u - U_i}{h_n}\right) \Bigg/ \sum_{i=0}^{n} K\left(\frac{u - U_i}{h_n}\right)$$

in which K and $\{h_n; n = 0, 1, \ldots\}$ are a suitably selected Borel measurable kernel and a number sequence, respectively. The estimate has been already applied to Hammerstein systems, see [3].

We want $E\{Y_n^2\}$ to be finite in further considerations. For this reason, we assume that

$$E\{m^2(U_n)\} < \infty. \tag{6}$$

Observe that (6) is satisfied, e.g., for any bounded m as well as for any m such that $|m(u)| \le \alpha_1 + \alpha_2|u|$, any α_1 and α_2, provided that $E\{U_n^2\} < \infty$. In both cases as well as in the whole paper, the functional form of m is completely unknown.

All restrictions presented in this section, in particular (2)-(4) and (6), hold throughout the paper and will not be repeated in any lemma nor theorem.

For matrices $A = [a_{ij}]$ and $B = [b_{ij}]$, $A \le B$ means that $a_{ij} \le b_{ij}$ for all i and j. Moreover, $A^+ = [|a_{ij}|]$. A matrix is called stable if all its eigenvalues lie in the unit circle. Moreover, for simplicity of notation, U has the same distributions as U_n.

III. Convergence of the algorithm

The kernel function satisfies the following restrictions:

$$\sup_v |K(v)| < \infty, \tag{7}$$

$$\int |K(v)|\, dv < \infty, \tag{8}$$

$$vK(v) \to 0 \text{ as } |v| \to \infty. \tag{9}$$

while the positive number sequence is selected to meet the following ones:

$$h_n \to 0 \text{ as } n \to \infty, \tag{10}$$

$$nh_n \to \infty \text{ as } n \to \infty. \tag{11}$$

Theorem 1: Let

$$\bar{A} = [\sup_x |a_{ij}(x)|] \text{ be a stable matrix.} \tag{12}$$

Let the kernel function satisfy (7)-(9). If the nonnegative number sequence satisfies (10) and (11), then

$$\hat{m}_n(u) \to \alpha m(u) \text{ as } n \to \infty \text{ in probability}$$

at every point u at which f and m are continuous and $f(u) > 0$.

Proof: Let u be a point at which both f and m are continuous and $f(u) > 0$. Denote

$$\hat{\xi}_n(u) = \frac{1}{nh_n} \sum_i = 0^n Y_i + 1K\left(\frac{u - U_i}{h_n}\right)$$

and observe

$$\begin{aligned} E\hat{\xi}_n(u) &= \frac{1}{nh_n} \sum_i = 0^n E\left\{E\left\{Y_i + 1|U_i\right\} K\left(\frac{u - U_i}{h_n}\right)\right\} \\ &= \frac{\alpha}{nh_n} \sum_i = 0^n E\left\{m(U_i) K\left(\frac{u - U_i}{h_n}\right)\right\} \end{aligned}$$

which, by virtue of (10) and Lemma 4 in Appendix, converges to $\alpha m(u) f(u) \int K(v)dv$ as $n \to \infty$. ■

In turn

$$\begin{aligned} \text{var}\left[\hat{\xi}_n(u)\right] &= \frac{1}{n^2 h_n^2} \text{var}\left[\sum_i = 0^n Y_i + 1K\left(\frac{u - U_i}{h_n}\right)\right] \\ &= S_n(u) + T_n(u) \end{aligned}$$

with

$$S_n(u) = \frac{1}{n^2 h_n^2} \sum_{i=0}^{n} \text{var}\left[Y_i + 1K\left(\frac{u - U_i}{h_n}\right)\right]$$

and

$$\begin{aligned} T_n(u) = \frac{1}{n^2 h_n^2} \sum_{i=0}^{n} \sum_{j=0, j \neq i}^{n} \text{cov}\left[Y_i + 1K\left(\frac{u - U_i}{h_n}\right),\right. \\ \left. Y_j + 1K\left(\frac{u - U_j}{h_n}\right)\right]. \end{aligned}$$

Using (16) in Appendix, we find

$$\begin{aligned} &\text{var}\left[Y_i + 1K\left(\frac{u - U_i}{h_n}\right)\right] \le E\left\{Y_i + 1^2 K^2\left(\frac{u - U_i}{h_n}\right)\right\} \\ &= E\left\{E\left\{Y_i + 1^2|U_i\right\} K^2\left(\frac{u - U_i}{h_n}\right)\right\} \\ &\le \beta E\left\{(1 + |m(U)|)^2 K^2\left(\frac{u - U}{h_n}\right)\right\} \end{aligned}$$

which leads to

$$\begin{aligned} nh_n S_n(u) &\le \frac{\beta}{h_n} E\left\{(1 + |m(U)|)^2 K^2\left(\frac{u - U}{h_n}\right)\right\} \\ &\le \beta \sup_h > 0 \frac{1}{h}\left|E\left\{(1 + |m(U)|)^2 K^2\left(\frac{u - U}{h}\right)\right\}\right|. \end{aligned}$$

Applying Lemma 4, we find the obtained quantity finite and come to a conclusion that $S_n(u) = O(1/nh_n)$. Recalling that $\bar{A}$ is stable, we find $\sum_{j=0}^{n} \bar{A}^j$ convergent as $n \to \infty$ and then, making use of Lemma 2 in Appendix, obtain

$$T_n(u) \le \frac{\rho(u)}{n^2 h_n} \sum_{i=0}^{n} \sum_{j=0, j \neq i}^{n} (c^+)^T \bar{A}^i - jPc^+ = O\left(\frac{1}{nh_n}\right).$$

Finally $\text{var}[\hat{\xi}_n(u)] = O(1/nh_n)$. In this way, we have shown that $\hat{\xi}_n(u) \to \alpha m(u) f(u) \int K(v)dv$ as $n \to \infty$ in probability. For similar reasons, $\hat{\eta}_n(u) \to f(u) \int K(v)dv$ as $n \to \infty$ in probability, where

$$\hat{\eta}_n(u) = \frac{1}{nh_n} \sum_{i=0}^{n} K\left(\frac{u - U_i}{h_n}\right).$$

Observing that $\hat{m}_n(u) = \hat{\xi}_n(u)/\hat{\eta}_n(u)$, we complete the proof.

Using similar arguments and applying Lemma 3 rather than Lemma 2, one can verify our next result:

Theorem 2: The assertion of Theorem 1 holds also with (12) replaced by $\sup_x \|A(x)\| < 1$ with the Euclidean norm.

Arguing as in [3] one can verify that, at every point u at the neighborhood of which both f and m have two derivatives,

$$|\hat{m}_n(u) - \alpha m(u)| = O\left(n^{-2/5}\right) \text{ in probability,}$$

provided that $h_n \sim n^{-1/5}$, $\int K(v)dv = 0$, and $\int v^2 K(v)dv < \infty$.

IV. Examples

We now present some systems in which the nonlinearity m can be recovered with the presented method. In the examples, $V_i = 0$, for $i = 0, -1-, 2, \ldots$, and $U_i = 0$, for $i = -1, -2, \ldots$.

Example 1: A first order system is described by the following difference equation:

$$X_n + a(X_{n-1})X_{n-1} = bm(U_{n-1}).$$

Restriction (12) is satisfied for $\sup_{-\infty < x < \infty} |a(x)| < 1$.

Example 2: Consider a system described by the following difference equation:

$$V_n + a_1(V_{n-1})V_{n-1} + a_0(V_{n-2})V_{n-2} = b_1 m(U_{n-1})$$

with $a_1(.)$ and $a_2(.)$ being even functions. Denoting $X_n = [V_{n-1}, V_n]^T$, we get (1) with

$$A(X_n) = \begin{bmatrix} 0 & 1 \\ -a_0(V_{n-1}) & -a_1(V_n) \end{bmatrix},$$

$b = [0, b_1]^T$, $c = [0, 1]^T$. Now

$$\bar{A} = \begin{bmatrix} 0 & 1 \\ \alpha_0 & \alpha_1 \end{bmatrix}$$

with $\alpha_i = \sup_v |a_i(v)|$, $i = 0, 1$. For $1 - \alpha_0 - \alpha_1 > 0$, $\alpha_0 + 1 > 0$, and $\alpha_1 - \alpha_0 + 1 > 0$, $\bar{A}$ is a stable matrix.

Example 3: A system of kth order is described by the following equation:

$$\begin{aligned} &V_n + a_{k-1}(V_{n-1})V_{n-1} + \cdots + a_0(V_{n-k})V_{n-k} \\ &= b_{k-1} m(U_{n-1}). \end{aligned}$$

For $X_n = [V_{n-k+1}, \cdots, V_{n-1}, V_n]^T$, we get (1) with

$$A(X_n) = \begin{bmatrix} 0 & I \\ -a^T(X_n) & \end{bmatrix},$$

$b = [0, \ldots, 0, b_{k-1}]^T$, and $c = [0, \ldots, 0, 1]^T$, where $a(X_n) = [a_0(V_{n-k+1}), \cdots, a_{k-1}(V_n)]^T$. Denoting $\alpha_i = \sup_v |a_i(v)|$, we obtain

$$\bar{A} = \begin{bmatrix} 0 & I \\ \alpha^T & \end{bmatrix},$$

where $\alpha^T = [\alpha_0, \cdots, \alpha_{k-1}]$. The stability of $\bar{A}$ can be easily verified with, e.g., the Schur-Cohn criterion, see [8].

Example 4: A system is described by the following equation:

$$\begin{aligned} &V_n + a_{k-1}(V_{n-1})V_{n-1} + \cdots + a_0(V_{n-k})V_{n-k} \\ &= b_{k-1} m(U_{n-1}) + \cdots + b_1 m(U_{n-k+1}) + b_0 m(U_{n-k}). \end{aligned}$$

Denoting by $X_n^{(i)}$ the ith coordinate of X_n, defining $X_n^{(1)} = b_0 U_{n-1} - a_0(V_{n-1})V_{n-1}$ and the other coordinates in the following recursive way: $X_n^{(i)} = X_{n-1}^{(i-1)} + (b_{i-1}U_{n-1} - a_{i-1}(V_{n-1})V_{n-1})$, $i = 2, \ldots, k$, we find

$$A(X_n) = \begin{bmatrix} 0 & -a(X_n) \\ I & \end{bmatrix},$$

$b = [b_0, \ldots, b_{k-1}]^T$, $c = [0, \ldots, 0, 1]^T$ with $a(X_n)$ as in the previous example.

V. Simulation example

The system is described by the following equation, see also Example 2:

$$V_n - 0.8V_{n-1}\cos(2V_{n-1}) + 0.2V_{n-2}\cos(3V_{n-2}) = m(U_{n-1})$$

with m shown in Fig. 2. The input signal U_n is uniformly distributed in the interval $(-2.5, 2.5)$ while Z_n has a Gaussian distribution with zero mean and variance 1. We have selected $K(u) = 1/(1+u^2)$ and $h_n = 3.5n^{-0.7}$. The mean integrated square error (MISE for short) defined as $\int_{-2}^{2} E(\hat{m}_n(u) - m(u))^2 du$ versus n is shown in Fig. 3. Results suggest that the required number of observations can be counted in hundreds.

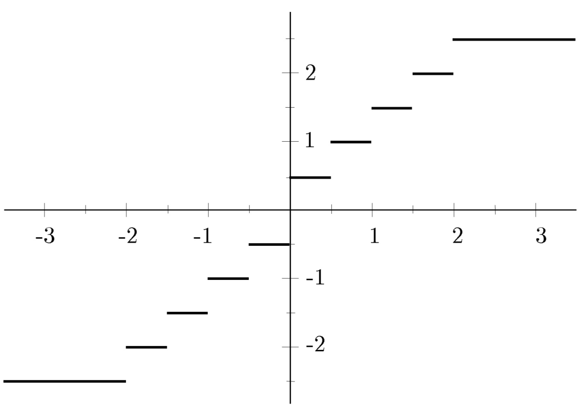

Fig. 2. The nonlinear characteristic m.

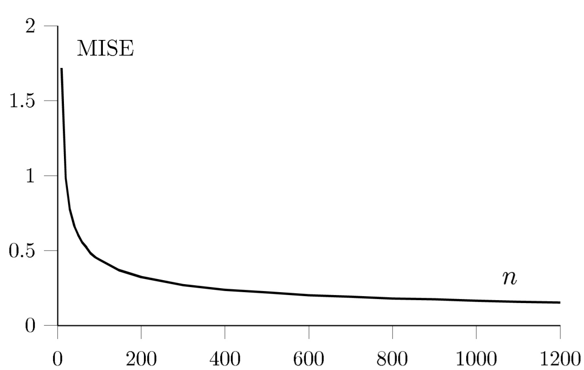

Fig. 3. MISE versus n.

VI. Final Remarks

The note enlarges the class of systems which input nonlinearity can be recovered under nonparametric *a priori* information. Dynamic subsystems may have not only a linear but also nonlinear transition matrix.

Appendix

A. Covariance

In the proof of Lemma 1, we need some inequalities. From (12) it follows that $X_{n+1}^+ \le \bar{A}X_n^+ + b^+|m(U_n)|$. Thus, since $E\{|m(U)|\} < \infty$, there exists a vector ξ such that

$$E\{X_n^+\} \le \xi \tag{13}$$

for all n. Moreover, $E\{X_{n+1}^+|U_n\} \le \bar{A}\xi + b^+|m(U_n)|$. Hence

$$E\{X_{n+1}^+|U_n\} \le d(1 + |m(U_n)|) \tag{14}$$

with some nonnegative vector d. Denoting $\Sigma_n = E\{(X_n X_n^T)^+\}$, we observe $\Sigma_{n+1} \le \bar{A}\Sigma_n\bar{A}^T + P$, where $P = 2\bar{A}\xi(b^+)^T E\{|m(U)|\} + (bb^T)^+ E\{m^2(U)\}$. Thus, since $\bar{A}$ is a stable matrix, there exists a matrix Σ such that $\Sigma_n \le \Sigma$ for all n. Therefore

$$\begin{aligned} &E\left\{(X_{n+1}X_{n+1}^T)^+|U_n\right\} \\ &\le \bar{A}\Sigma\bar{A}^T + 2\bar{A}\xi(b^+)^T|m(U_n)| + (bb^T)^+m^2(U_n) \\ &\le Q(1 + |m(U_n)|)^2 \end{aligned} \tag{15}$$

with some matrix Q. From this and the fact that $\text{var}[Z_n] < \infty$, we conclude that

$$E\left\{Y_{n+1}^2|U_n\right\} \le \beta(1 + |m(U_n)|)^2 \tag{16}$$

with some β.

Lemma 1: Let (12) hold. For $i > j$,

$$\begin{aligned} &(\text{cov}\,[X_{i+1}\varphi(U_i), X_{j+1}\varphi(U_j)])^+ \\ &\le \bar{A}^{i-j}SE\left\{(1 + |m(U)|)^2\,|\varphi(U)|\right\}E\{|\varphi(U)|\} \end{aligned}$$

with a matrix S independent of both i and j.

Proof: In the first step, we verify the following equality:

$$\begin{aligned} &\text{cov}\,[X_i + 1\varphi(U_i), X_j + 1\varphi(U_j)] \\ &= E\{\varphi(U)\}\text{cov}\,[A(X_i)\cdots A(X_j+1)X_j + 1, X_j + 1\varphi(U_j)]\,. \end{aligned} \tag{17}$$

Since $X_{i+1} = A(X_i)X_i + bm(U_i)$ and U_i is independent of both U_j and X_{j+1}, the examined covariance equals

$$\begin{aligned} &\text{cov}\,[A(X_i)X_i\varphi(U_i), X_j + 1\varphi(U_j)] \\ &= E\{\varphi(U)\}\text{cov}\,[A(X_i)X_i, X_j + 1\varphi(U_j)]\,. \end{aligned}$$

For similar reasons, we find the covariance in the last expression equal to

$$\begin{aligned} &\text{cov}\,[A(X_i)A(X_i-1)X_i - 1, X_j + 1\varphi(U_j)] \\ &= \text{cov}\,[A(X_i)A(X_i-1)\cdots A(X_j+1)X_j + 1, X_j + 1\varphi(U_j)] \end{aligned}$$

and prove (17). To begin the second part of the proof, observe that the covariance in (17) equals $R_1 - R_2$ with

$$R_1 = E\left\{A(X_i)A(X_i-1)\cdots A(X_j+1)X_j + 1X_j + 1^T\varphi(U_j)\right\}$$

and

$$\begin{aligned} R_2 = &E\{A(X_i)A(X_i-1)\cdots A(X_j+1)X_j + 1\} \\ &E\left\{X_j + 1^T\varphi(U_j)\right\} \end{aligned}$$

Using (15), we get

$$\begin{aligned} R_1^+ &\le \bar{A}^i - jE\left\{(X_j + 1X_j + 1^T)^+|\varphi(U_j)|\right\} \\ &= \bar{A}^i - jE\left\{E\left\{(X_j + 1X_j + 1^T)^+|U_j\right\}|\varphi(U_j)|\right\} \\ &\le \bar{A}^i - jQE\left\{(1 + |m(U)|)^2\,|\varphi(U)|\right\}. \end{aligned}$$

Since (13) and (14) lead to

$$\begin{aligned} R_2^+ &\leq \bar{A}^i - jE\{X_j + 1^+\}E\left\{E\{(X_j + 1^T)^+|U_j\}\varphi(U_j)|\right\} \\ &\leq \bar{A}^i - j\xi d^T E\left\{(1 + |m(U)|)|\varphi(U)|\right\}, \end{aligned}$$

we thus find $(R_1 - R_2)^+ \leq \bar{A}^{i-j}SE\left\{(1 + |m(U)|)^2 |\varphi(U)|\right\}$ with some matrix S independent of both i and j. Recalling (17), we easily complete the proof. ■

Lemma 2: Let (12) hold. Let u be a point at which both m and f are continuous. For $i > j$,

$$\begin{aligned} &\left|\text{cov}\left[Y_{i+1}K\left(\frac{u - U_i}{h}\right), Y_{j+1}K\left(\frac{u - U_j}{h}\right)\right]\right| \\ &\leq h\rho(u)(c^+)^T\bar{A}^{i-j}Pc^+ \end{aligned}$$

with some $\rho(u) < \infty$ and some matrix P independent of i, j, and h.

Proof: It suffices to replace $\varphi(U_i)$ with $K((u - U_i)/h)$ in Lemma 2 and then use Lemma 4. ■

The proof of our next lemma is straightforward and will be omitted.

Lemma 3: Let $\|A\| < 1$. Let u be a point at which both m and f are continuous. For $i > j$,

$$\left|\text{cov}\left[Y_{i+1}K\left(\frac{u - U_i}{h}\right), Y_{j+1}K\left(\frac{u - U_j}{h}2\right)\right]\right| \leq h\theta(u)\gamma^{i-j}$$

with $\theta(u) < \infty$ independent of i, j, and h, where $\gamma = \sup_x \|A(x)\|$.

B. Lemma

The following result can be found in [12, Theorems 9.9 and 9.13].

Lemma 4: Let f be a density function of a random variable U. Let ϕ be a Borel function such that $E|\phi(U)| < \infty$. If a Borel measurable kernel K satisfies (7)–(9), then

$$\frac{1}{h}E\left\{\phi(U)K\left(\frac{u - U}{h}\right)\right\} \to \phi(u)f(u)\int K(v)dv \text{ as } h \to 0$$

and

$$\sup_{h>0}\frac{1}{h}E\left\{\left|\phi(U)K\left(\frac{u - U}{h}\right)\right|\right\} < \varphi(u)$$

with some $\varphi(u)$ finite at every point u at which both ϕ and f are continuous.

ACKNOWLEDGEMENTS

The author would like to thank to reviewers for their comments and suggestions.

REFERENCES

[1] S. A. Billings and S. Y. Fakhouri, "Identification of systems containing linear dynamic and static nonlinear elements," *Automatica*, vol. 18, no. 1, pp. 15–26, 1982.

[2] E. Eskinat, S. H. Johnson, and W. L. Luyben, "Use of Hammerstein models in identification of nonlinear systems," *American Institute of Chemical Engineers Journal*, vol. 37, no. 2, pp. 255–268, 1991.

[3] W. Greblicki and M. Pawlak, "Identification of discrete Hammerstein systems using kernel regression estimates," *IEEE Transactions on Automatic Control*, vol. AC-31, no. 1, pp. 74–77, January 1986.

[4] ——, *Nonparametric System Identification*. Cambridge: Cambridge University Press, 2008.

[5] Z. Hasiewicz, "Non-parametric estimation of non-linearity in a cascade time-series system by multiscale approximation," *Signal Processing*, vol. 81, pp. 791–807, 2001.

[6] M. J. Korenberg and I. W. Hunter, "The identification of nonlinear biological systems: LNL cascade models," *Biological Cybernetics*, vol. 55, pp. 125–134, 1986.

[7] A. Krzyżak, "On estimation of a class of nonlinear systems by the kernel regression estimate," *IEEE Transactions on Information Theory*, vol. 36, no. 1, pp. 141–152, February 1990.

[8] P. Lancaster and M. Tismenetsky, *The Theory of Matrices*. San Diego: Academic Press, 1985.

[9] Y. Liu and E. W. Bai, "Iterative identification of Hammerstein systems," *Automatica*, vol. 43, no. 2, pp. 346–354, 2007.

[10] S. W. Su, L. Wang, B. G. Celler, and A. W. Savkin, "Oxygen uptake estimation in humans during exercise using a Hammerstein model," *Annals of Biomedical Engineering*, vol. 35, pp. 1898–1906, November 2007.

[11] S. W. Su, L. Wang, B. G. Celler, A. W. Savkin, and Y. Guo, "Identification and control for heart rate regulation during treadmill exercise," *IEEE Transactions on Biomedical Engineering*, vol. 54, pp. 1238–1246, July 2007.

[12] R. L. Wheeden and A. Zygmund, *Measure and Integral*. New York: Dekker, 1977.